PHYS 102 – General Physics II Final Exam

Duration: 120 minutes Tuesday, 27 December 2022; 18:30

1 . A ring of radius R_1 is centered at the origin and carries an increasing current $I(t) = \alpha t$, $(\alpha > 0)$, counterclockwise direction in the *xy* plane as shown in the figure. A second, much smaller loop of radius R_2 is placed parallel to the first loop with its center at $z = h$.

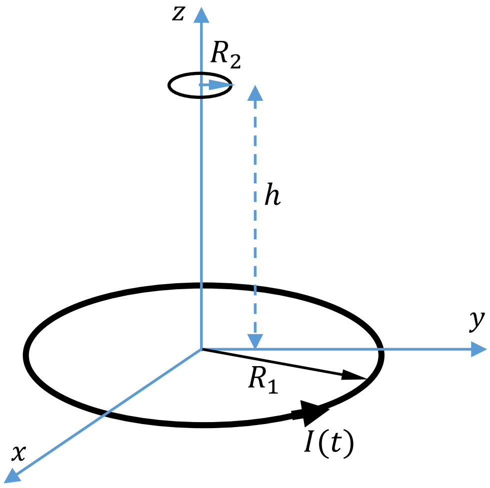
(a) (3 Pts.) What is the magnetic moment (vector) of the bottom loop?

(b) (8 Pts.) What is the magnitude of the magnetic field created by the bottom loop at the center of the top loop?

(c) (6 Pts.) Assuming that the top loop is far away ($R_2 \ll h$, and small ($R_2 \ll R_1$), what is the mutual inductance between the loops?

(d) (4 Pts.) In which direction is the induced current in the small loop? (Same as the first loop or opposite?, Explain your reasoning)

(e) (4 Pts.) Is the large loop attracting or repelling the second loop, why?



Solution: (a) $\vec{\mu} = IA\,\hat{\mathbf{k}} \quad \rightarrow \quad \vec{\mu} = \pi R_1^2 \alpha t\,\hat{\mathbf{k}}$.

(b) Magnetic field produced by the bottom ring at $z = h$ is found as

$$B_x = B_y = 0\,, \qquad dB_z = \frac{\mu_0}{4\pi}\frac{Id\ell}{R_1^2 + h^2}\frac{R_1}{\sqrt{R_1^2 + h^2}} \quad \rightarrow \quad B_z = \frac{\mu_0 I R_1}{4\pi(R_1^2 + h^2)^{3/2}}\int d\ell = \frac{\mu_0 \alpha R_1^2 t}{2(R_1^2 + h^2)^{3/2}}\,.$$

(c) Since $R_2 \ll R_1$, and $R_2 \ll h$ we assume that the magnitude of the magnetic field is constant over the area of the top loop.

$$M = \frac{N_2 \Phi_2}{I_1} \quad \rightarrow \quad M = \frac{\pi R_2^2 B_z}{I} \quad \rightarrow \quad M = \frac{\mu_0 \pi R_1^2 R_2^2}{2(R_1^2 + h^2)^{3/2}}\,.$$

(d) Increasing current in the bottom loop couses the flux through the top loop to increase. By Lenz's law, the current in the top loop should be in the **opposite direction** to oppose the increase.

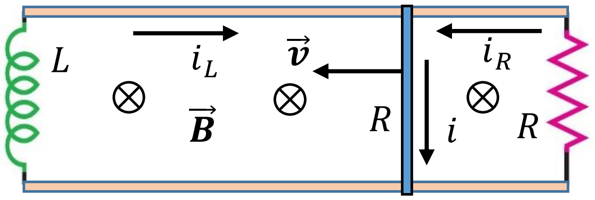
(e) Currents being in opposite directions causes the force to be **repulsive**.

2. A conducting rod with resistance R is free to slide on two parallel conducting rails of negligible resistance, separated a distance ℓ. An inductor of inductance L and a resistor of resistance R are connected across the ends of the rails to form a loop. A constant uniform magnetic field of magnitude B directed perpendicularly into the page exists everywhere inside the loop. Starting at time $t = 0$, when there is no current is in the inductor, an external force starts pulling the rod to the left at a constant speed v. Find:

(a) (5 Pts.) direction of the current in the rod;

(b) (10 Pts.) magnitude of the current in L as a function of time;

(c) (10 Pts.) instantaneous power supplied by the applied force that is needed to move the rod at constant speed.



Solution: (a) Motional emf $\mathcal{E} = B\ell v$ causes a current i in the rod, **downward** in direction.

(b) Current i_L in the inductor and current i_R in the resistor should satisfy the junction rule $i = i_R + i_L$. Applying the loop rule to the two loops, we have

$$B\ell v - (i_R + i_L)R - L\frac{di_L}{dt} = 0\,, \qquad B\ell v - (i_R + i_L)R - Ri_R = 0 \;\rightarrow\; i_R = \frac{L}{R}\frac{di_L}{dt}\,.$$

Using the last equation in the first one, we get

$$B\ell v - Ri_L - 2L\frac{di_L}{dt} = 0 \;\rightarrow\; \frac{di_L}{i_L - B\ell v/R} = -\left(\frac{R}{2L}\right)dt \;\rightarrow\; i_L(t) = \frac{B\ell v}{R}\left(1 - e^{-Rt/2L}\right).$$

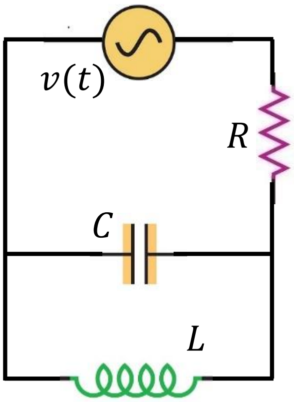
Hence,

$$i_R = \frac{L}{R}\frac{di_L}{dt} \;\rightarrow\; i_R(t) = \frac{B\ell v}{2R}e^{-Rt/2L}\,.$$

(c) To make the rod move at constant velocity, the magnitude of the applied force must be equal to the magnitude of the magnetic force on the rod.

$$F = (i_R + i_L)B\ell = \frac{B^2\ell^2 v}{R}\left(1 - \frac{1}{2}e^{-Rt/2L}\right) \;\rightarrow\; P = Fv = \frac{B^2\ell^2 v^2}{R}\left(1 - \frac{1}{2}e^{-Rt/2L}\right).$$

3. Consider the circuit shown in the figure, where the ac source voltage is given by $v(t) = V_0 \sin(\omega t)$.



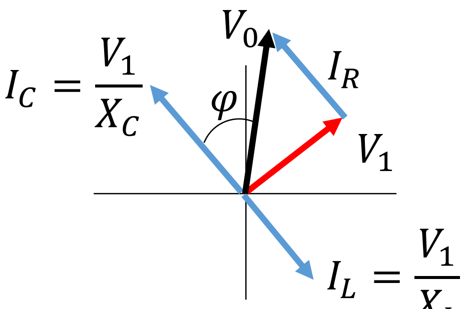
(a) (8 Pts.) What is the total impedance?

(b) (10 Pts.) Find the current $i_R(t)$ through the resistor.

(c) (7 Pts.) What is the average power dissipated in the circuit?

Solution:

Voltage across the capacitor is equal to the voltage across the inductor, because they are connected in parallel. Denoting by V_1 the amplitude of this voltage, let us draw the phasor diagram assuming $X_C < X_L$. The formulas will be valid for $X_C > X_L$ as well. We have the following relations:



(a)

$$I_R = I_C - I_L = \left(\frac{1}{X_C} - \frac{1}{X_L}\right) V_1 = \left(\omega C - \frac{1}{\omega L}\right) V_1 \, ,$$

$$V_R = R I_R \quad \rightarrow \quad V_R = R\left(\omega C - \frac{1}{\omega L}\right) V_1 \, ,$$

$$V_0^2 = V_1^2 + V_R^2 = V_1^2 \left[1 + R^2\left(\omega C - \frac{1}{\omega L}\right)^2\right].$$

This means

$$V_1 = \frac{V_0}{\sqrt{1 + R^2\left(\omega C - \frac{1}{\omega L}\right)^2}} \quad \rightarrow \quad I_R = \frac{\left(\omega C - \frac{1}{\omega L}\right) V_0}{\sqrt{1 + R^2\left(\omega C - \frac{1}{\omega L}\right)^2}} = \frac{V_0}{\sqrt{R^2 + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^{-2}}}.$$

$$Z = \frac{V_0}{I_R} \quad \rightarrow \quad Z = \sqrt{R^2 + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^{-2}} \, , \qquad \tan\varphi = \frac{V_1}{V_R} = \frac{1}{R\left(\omega C - \frac{1}{\omega L}\right)} = \frac{1}{R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)} = \frac{X_L X_C}{R(X_L - X_C)}.$$

(b) $$i_R(t) = I_R \sin(\omega t + \varphi) \quad \rightarrow \quad i_R(t) = \frac{V_0}{\sqrt{R^2 + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^{-2}}} \sin(\omega t + \varphi) \, .$$

(c) $$P_{\mathrm{av}} = \frac{1}{2} R I_R^2 \quad \rightarrow \quad P_{\mathrm{av}} = \frac{R}{2} \frac{V_0^2}{R^2 + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^{-2}} = \frac{R}{2} \frac{V_0^2 \left(\omega C - \frac{1}{\omega L}\right)^2}{1 + R^2\left(\omega C - \frac{1}{\omega L}\right)^2}.$$

4. A capacitor is made of two parallel concentric circular plates of radius R, and is always connected to an ideal battery of V_0 volts. At time $t = 0$, the distance between the plates of the capacitor is d_0 and, decreases with time as

$d(t) = d_0 e^{-\beta t}$, where $\beta > 0$.

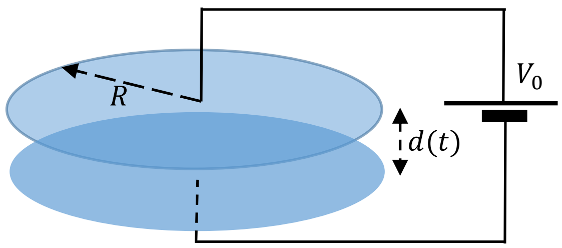
Assume that the distance between the plates is always much smaller than the radius so that fringing can be neglected.

(a) (5 Pts.) What is the magnitude of the electric field in the capacitor as a function of time?

(b) (6 Pts.) What is the magnitude of the magnetic field between the plates as a function of distance from the symmetry axis?

(c) (7 Pts.) What is the total electrical energy stored in the capacitor as a function of time?

(d) (7 Pts.) Calculate the Poynting vector (both direction and magnitude) between the plates.



Solution: (a)

$$|\vec{\mathbf{E}}(t)| = E(t) = \frac{V_0}{d(t)} \quad \rightarrow \quad E(t) = \frac{V_0}{d_0} e^{\beta t}.$$

(b) Applying the Ampére-Maxwell law to a circular path of radius r centered at the axis of symmetry, we have

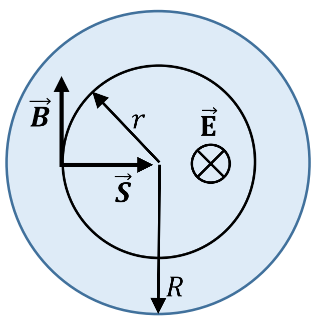
$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad \rightarrow \quad 2\pi r B(r) = \mu_0 \epsilon_0 \pi r^2 \beta \frac{V_0}{d_0} e^{\beta t} \quad \rightarrow \quad B(r) = \frac{\mu_0 \epsilon_0 \beta V_0}{2d_0} r e^{\beta t}.$$

(c)

$$U_E = \left(\frac{1}{2}\epsilon_0 E^2\right)\left(\pi R^2 d(t)\right) \quad \rightarrow \quad U_E = \frac{1}{2}\left(\frac{\epsilon_0 \pi R^2}{d_0 e^{-\beta t}}\right) V_0^2 = \frac{1}{2} C(t) V_0^2 .$$

(d)

$$S = \frac{1}{\mu_0}|\vec{\mathbf{E}} \times \vec{\mathbf{B}}| \quad \rightarrow \quad S = \frac{\epsilon_0 V_0^2 \beta}{2d_0} r e^{2\beta t}.$$



Magnitude of the electric field is increasing in time means the displacement current is in the direction of the electric field. This produces a magnetic field which is in the clockwise direction, as shown in the figure. So, the direction of the Poynting vector is inward towards the center.